Losing your Groverhead: new upper bounds and Q# implementations of AES and LowMC

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Overview

- AES
- Grover's search
- Quantum circuits
- Previous work
- Our improvements
- LowMC
- Future directions

- AES [DR01] is a block cipher standardized by NIST in '01.
- Key lengths in $\{128, 192, 256\}$ bits, block size 128 bits.

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- AES-128 (resp. -192, -256) uses 10 (resp. 12, 14) rounds.

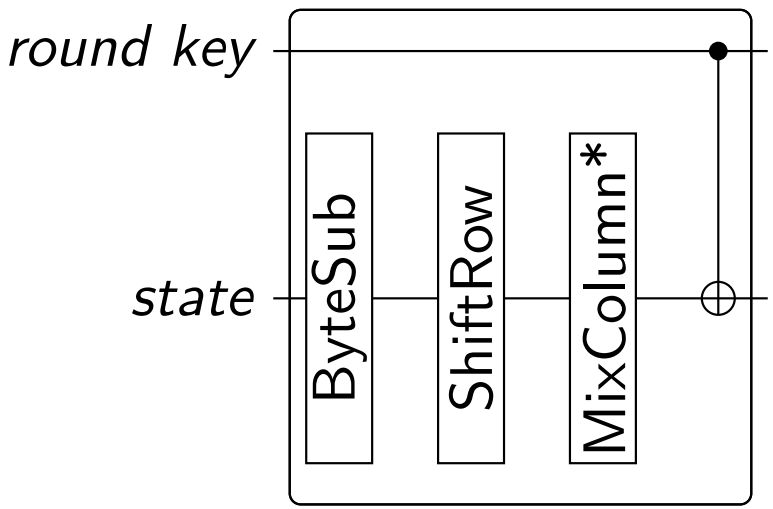

Figure: AES round design.

- The round key is expanded from the ciphers' key, and 128 bits are XOR'd on the state each round.

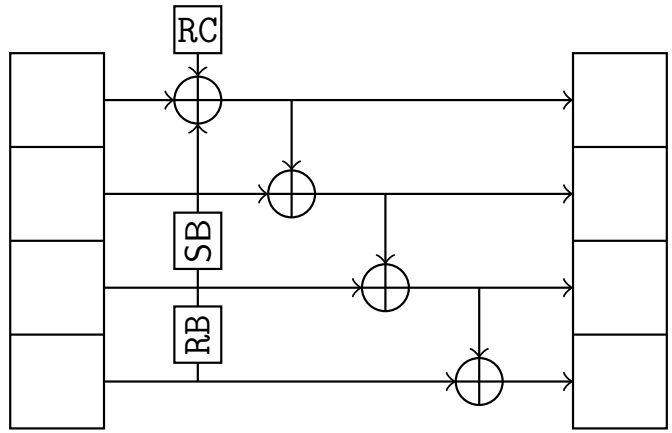

Figure: AES-128 key expansion 'round'. Credit: Jérémy Jean.

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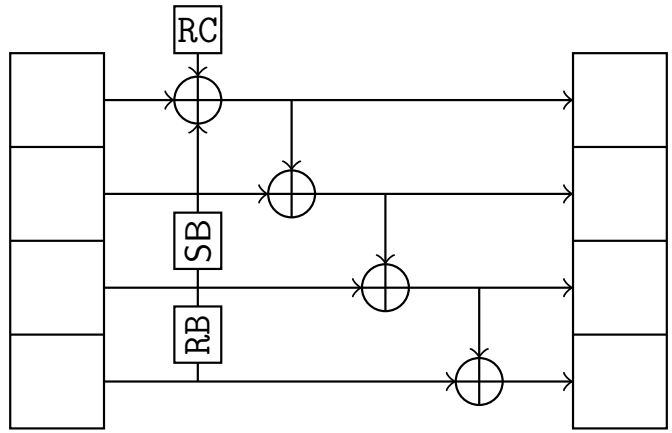
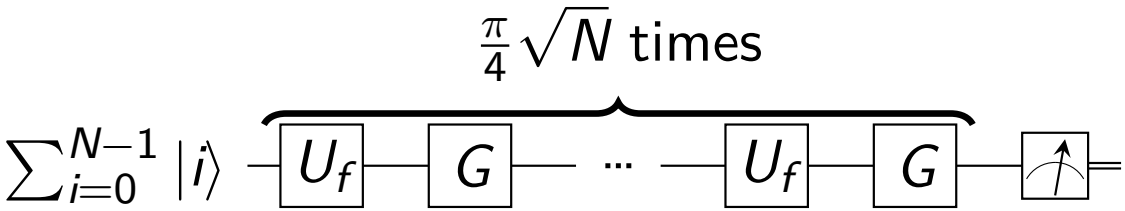


Figure: AES-128 key expansion 'round'. Credit: Jérémy Jean.

- ShiftRow and RotByte are permutations.
- MixColumn is an invertible linear transformation.
- ByteSub and SubByte are byte-wise applications of the S-box, which computes inversion in $\mathbb{F}_{2^8}$ (and maps $0 \mapsto 0$).

Grover's search [Gro96] is a quantum algorithm for finding elements in unsorted lists of size N.



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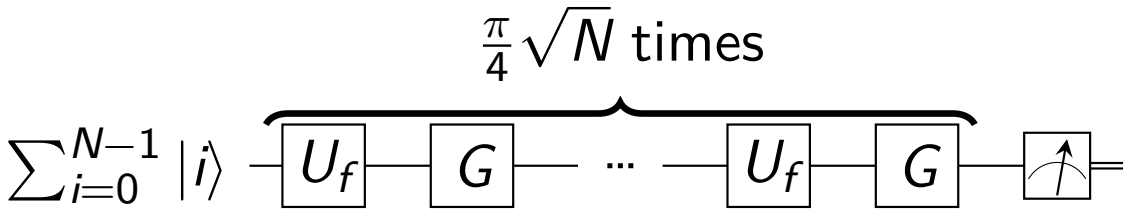

Figure: Grover's search sketch.

- U_f is a quantum circuit mapping $|x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$, where $f(x) = (x == target)$.
- G is a "reflection around the mean" operation.

- A quantum circuit is a sequence of unitary operators (and measurements).
- Width: the maximum number of qubits used.
- Depth: the number of sequential "basic" operations.
- Complex operations can be constructed from simple (universal) sets of gates. We use Clifford + T.

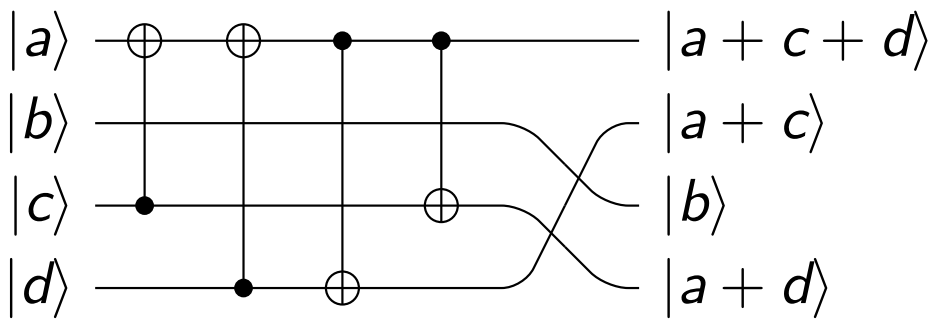

Figure: Quantum circuit example.

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This is not necessarily "realistic", but is what the previous literature on AES (and hence NIST in [Nat16]) uses.

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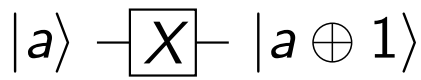
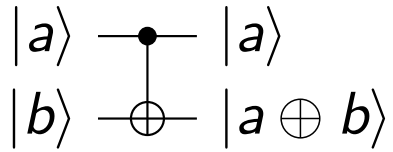


- CNOT gate, works similarly to an XOR gate .



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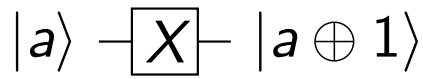

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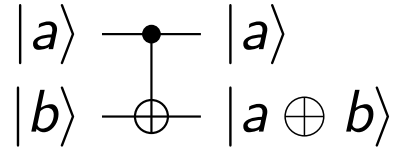
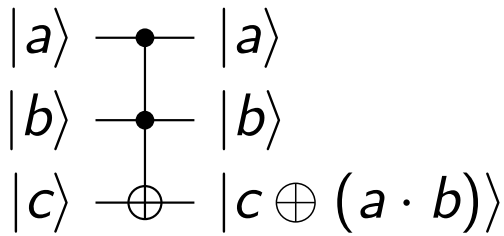


- Toffoli (aka CCNOT), works similarly to an AND gate.



Linear programs are sequences of $w \leftarrow x \star y$, for some binary operation $\star$.

t_1	$=$	$x_2 + x_3$	t_2	$=$	$x_2 \times x_0$	t_3	$=$	$x_1 + t_2$
t_4	$=$	$x_0 + x_1$	t_5	$=$	$x_3 + t_2$	t_6	$=$	$t_5 \times t_4$
t_7	$=$	$t_3 \times t_1$	t_8	$=$	$x_0 \times x_3$	t_9	$=$	$t_4 \times t_8$
t_{10}	$=$	$t_4 + t_9$	t_{11}	$=$	$x_1 \times x_2$	t_{12}	$=$	$t_1 \times t_{11}$
t_{13}	$=$	$t_1 + t_{12}$	y_0	$=$	$t_2 + t_{13}$	y_1	$=$	$x_3 + t_7$
y_2	$=$	$t_2 + t_{10}$	y_3	$=$	$x_1 + t_6$			

Figure 1: Inversion in $GF(2^4)$. Input is (x_0, x_1, x_2, x_3) and output is (y_0, y_1, y_2, y_3).

These can be easily translated into Q# by writing the appropriate self-inverse operators $(x, y, z) \mapsto (x, y, z \oplus (x \star y))$.

Image taken from [BP11].

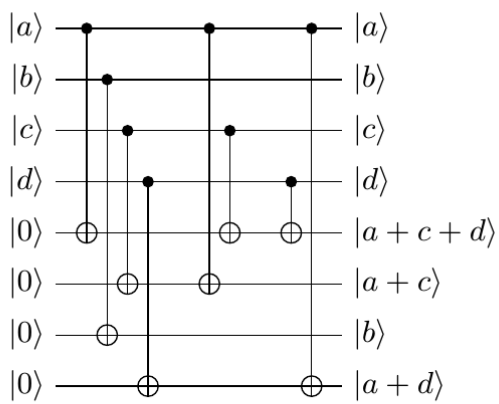
Multiplication by constant matrices:

- Naively, using ancilla qubits (b).
- Rearranging operations from the naive version **will help**.
- Invertible matrices can be inplemented in-place by PLU decomposing them [TB97] (c).

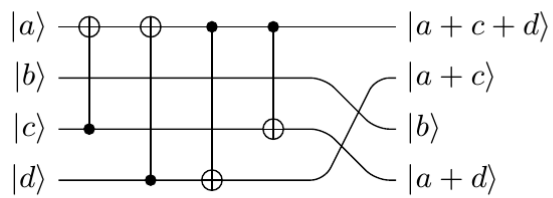
And the P is for free!

$$M = \begin{pmatrix} 1&0&1&1 \\ 1&0&1&0 \\ 0&1&0&0 \\ 1&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 1&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&1&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} = P \cdot L \cdot U$$

(a) Invertible linear transformation M and its PLU decomposition.



(b) Naive circuit computing M.



(c) In-place implementation of M.

Fig. 1. Alternative circuits implementing the same linear transformation $M : \mathbb{F}_2^4 \to \mathbb{F}_2^4$, by using the two strategies described in Section 2.3.

- [YI00] introduces the idea of using Grover's to attack block ciphers with keylength k in $O(2^{k/2})$ "operations" (U_f and G).
- Grassl et al. [GLRS16] provide the first cost estimate for U_f (and disregard G, we do the same!).

Component	[GLRS16] design
ShiftRow, RotByte	Rewiring (free)
MixColumn	Multiplication by constant in $\mathbb{F}_{2^8}[x]/(x^4+1)$: Invertible linear map, hence PLU decomposition
S-box	Inversion $\equiv ((\alpha \cdot \alpha^2) \cdot (\alpha \cdot \alpha^2)^4 \cdot (\alpha \cdot \alpha^2)^{16} \cdot \alpha^{64})^2$ in $\mathbb{F}_2[x]/\langle x^8 + x^4 + x^3 + x + 1\rangle$:
AddRoundKey	Bitwise XOR/CNOT
KeyExpansion	Caching of "expensive" bytes + recomputing of "cheap" ones
Rounds	"Pebbling" to reduce the number of ancilla qubits

When computing rounds, they try to tradeoff between circuit width and depth, using a "pebbling" strategy.

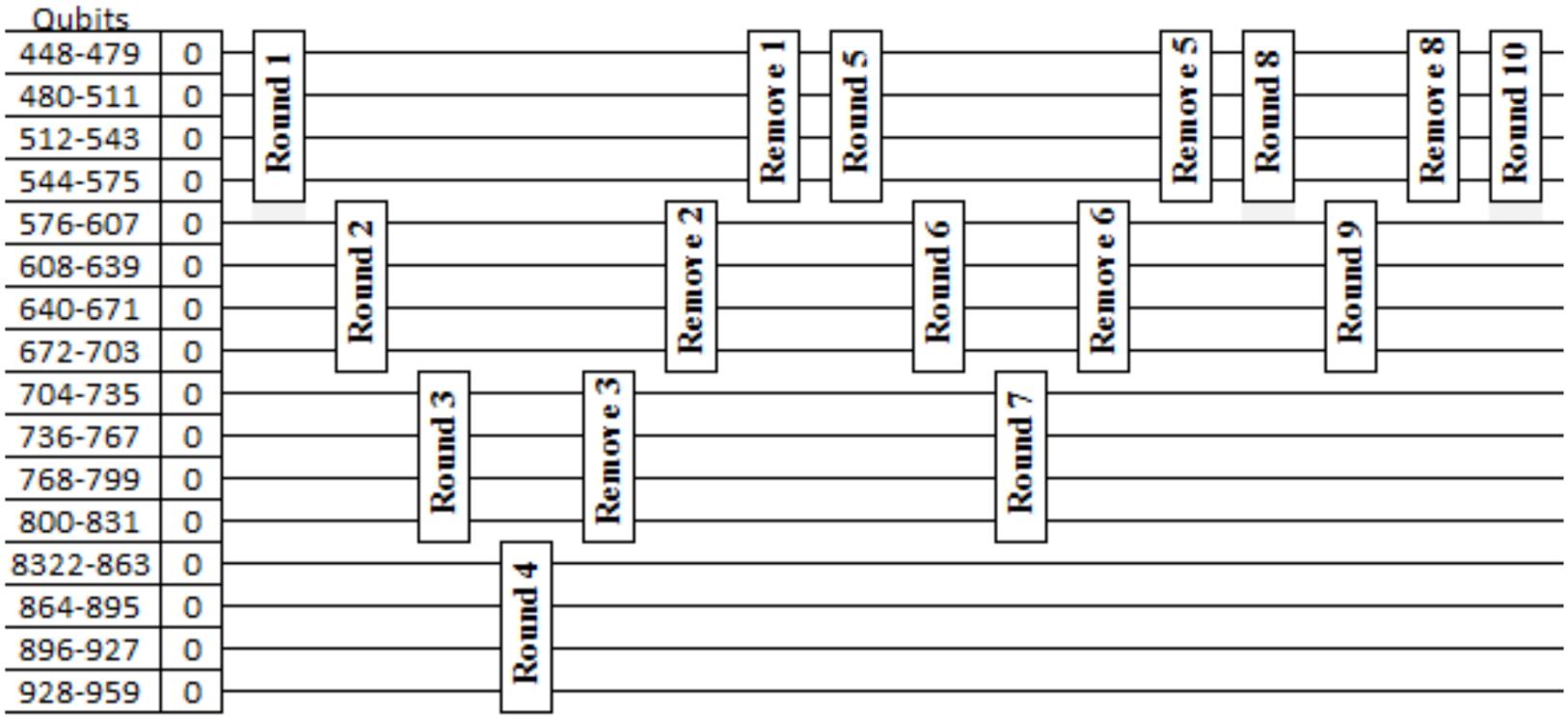


Figure: Round pebbling strategy for AES-128 in [GLRS16].

- Given any block cipher C: $\{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$, for any (m, c) pair there may be multiple K such that $C(K, m) = c$.
- Hence, when doing exhaustive key search, multiple (m_i, c_i) pairs may be needed to uniquely determine a key K whp.
- [GLRS16] find that for AES-128 (resp. -192, -256), 3 (resp. 4, 5) pairs are needed when implementing U_f.

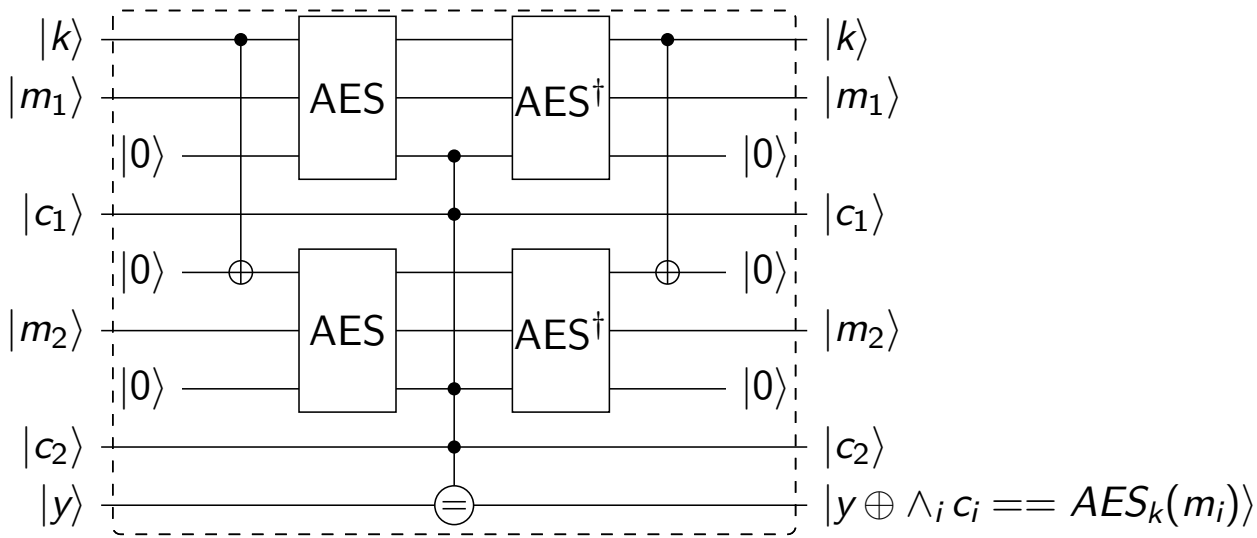


Figure: Example AES U_f for two (m_i, c_i) pairs.

Overall, Grassl et al. estimate the cost of U_f as follows.

scheme	#(1qCliff+CNOT)	#T	T-depth	full depth	width	G-cost	DW-cost
AES-128	$1.55 \cdot 2^{86}$	$1.19 \cdot 2^{86}$	$1.06 \cdot 2^{80}$	$1.16 \cdot 2^{81}$	2 953	$1.37 \cdot 2^{87}$	$1.67 \cdot 2^{92}$
AES-192	$1.17 \cdot 2^{119}$	$1.81 \cdot 2^{118}$	$1.21 \cdot 2^{112}$	$1.33 \cdot 2^{113}$	4 449	$1.04 \cdot 2^{120}$	$1.44 \cdot 2^{125}$
AES-256	$1.83 \cdot 2^{151}$	$1.41 \cdot 2^{151}$	$1.44 \cdot 2^{144}$	$1.57 \cdot 2^{145}$	6 681	$1.62 \cdot 2^{152}$	$1.28 \cdot 2^{158}$

Table: Circuit size for U_f as in [GLRS16].

- In 2016, NIST puts out a call for post-quantum cryptography proposals [Nat16].
- Regarding quantum computation capabilities, they suggest having a $\texttt{MAXDEPTH} \in \{2^{40}, 2^{64}, 2^{96}\}$ parameter bounding quantum computation depth.
- They also define security "categories" 1, 3, and 5, based on the hardness of key recovery against AES-128, -192, -256.

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- They also define security "categories" 1, 3, and 5, based on the hardness of key recovery against AES-128, -192, -256.
- Early termination of Grover's search results in low success probabilities.
- Hence, due to `MAXDEPTH`, Grover's search against AES needs to be parallelised.

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- Using D and G from [GLRS16], they deduce the security categories' requirements.

AES 128	2^{170}/MAXDEPTH quantum gates
AES 192	2^{233}/MAXDEPTH quantum gates
AES 256	2^{298}/MAXDEPTH quantum gates

Figure: Definition for security categories 1, 3, and 5.

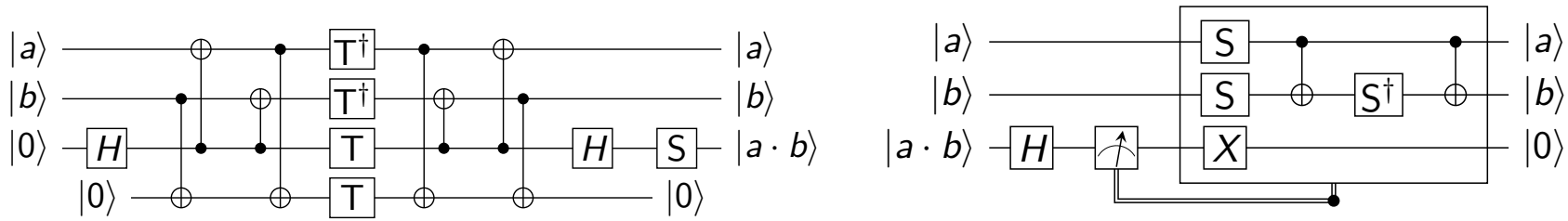
- Our initial idea: NIST cares about limiting depth, but uses [GLRS16] which optimizes for width. What if we minimize depth?
- Hindsight: parallelisation is bad, so crucially beneficial to minimise depth!
- We also get a Q# implementation:
  - testable,
  - friendly to read/modify,
  - automated circuit size estimates,
  - easy to translate linear programs/verilog using regexes!

We look now at our design choices for a smaller Grover oracle for AES.

S-box: well investigated in the hardware literature.

- Lots of linear programs to translate and test.
- Tried various variants of [BP11].
- Scooped! In concurrent indepedent work, Langenberg et al. [LPS19] propose a similar S-box change.
  - They keep the same pebbling strategy of [GLRS16] and provide only an implementation of their S-box.

- Inside their S-box, [GLRS16] use a 7 T-gates implementation of Toffoli.
- We replace Toffoli with AND gates, using a custom design by Mathias Soeken, based on Selinger [Sel13] and Gidney [Gid18].



(a) AND gate.

(b) AND† gate.

Figure: AND gate with T-depth 1, T count 4, and "T-free" adjoint operator. It does introduce measurements.

- KeyExpansion: instead of caching, we do in-place expansion as necessary.

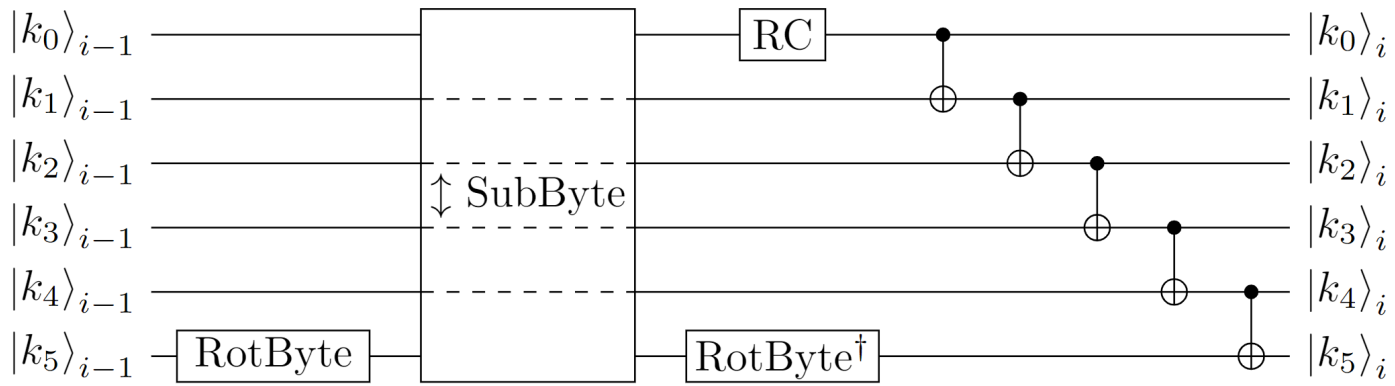


Figure: AES 192 in-place i^{th} round key expansion.

- This saves us qubits with respect to naive full expansion, while not increasing depth due to the computations running in parallel to the round.

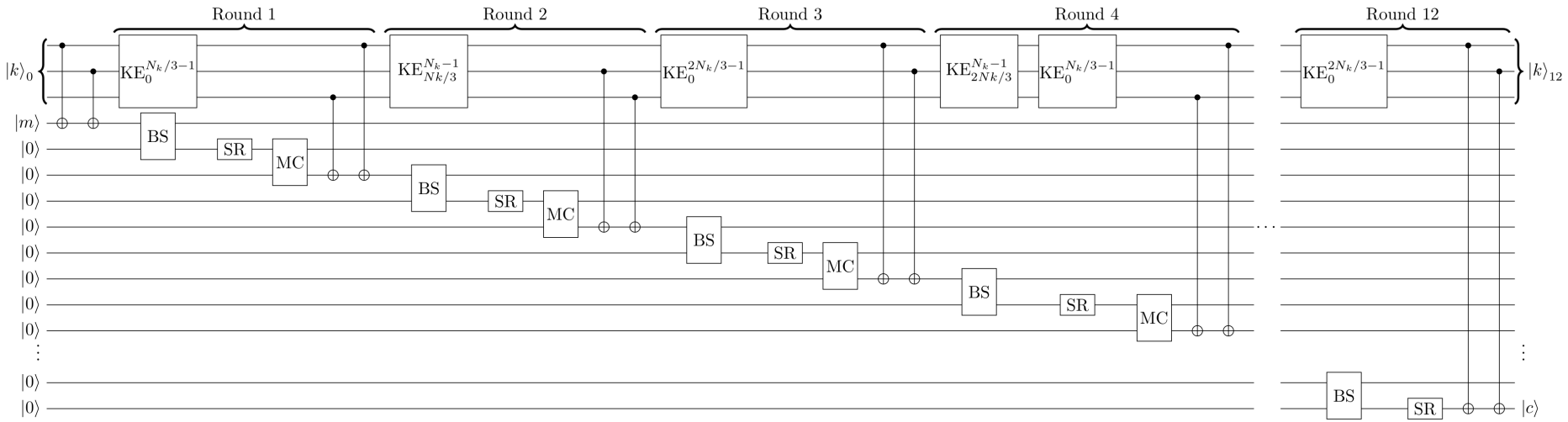


Figure: AES 192 round structure.

Indeed, rounds only require 128 bits of expanded key at every time.

Other improvements:

- We cost both the PLU-decomposed in-place MixColumn design, and a recent, shallower (but wider) design by Maximov [Max19].
- Fix to the key uniqueness computation: 3, 4, 5 pairs are too many!
  - For $p \approx 1$ attacks, 2, 2, 3 pairs are enough.
  - As Langenberg et al. [LPS19] also noticed, we suggest using 1, 2, 2 pairs for high probability attacks ($\approx 1/e$, ≈ 1, $\approx 1/e$) when using unbounded Grover's.

scheme	#(1qCliff+CNOT)	#T	#M	T-depth	full depth	width	*G*-cost	*DW*-cost	p_{succ}
Grassl et al. [GLRS16]									
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AES-256 ($r=5$)	$1.83 \cdot 2^{151}$	$1.41 \cdot 2^{151}$	0	$1.44 \cdot 2^{144}$	$1.57 \cdot 2^{145}$	6 681	$1.62 \cdot 2^{152}$	$1.28 \cdot 2^{158}$	≈ 1
Langenberg et al. [LPS19]									
AES-128 ($r=1$)	$1.46 \cdot 2^{82}$	$1.47 \cdot 2^{81}$	0	$1.44 \cdot 2^{77}$	$1.39 \cdot 2^{79}$	865	$1.10 \cdot 2^{83}$	$1.17 \cdot 2^{89}$	$\approx 1/e$
AES-192 ($r=2$)	$1.71 \cdot 2^{115}$	$1.68 \cdot 2^{114}$	0	$1.26 \cdot 2^{109}$	$1.23 \cdot 2^{111}$	1 793	$1.27 \cdot 2^{116}$	$1.08 \cdot 2^{122}$	≈ 1
AES-256 ($r=2$)	$1.03 \cdot 2^{148}$	$1.02 \cdot 2^{147}$	0	$1.66 \cdot 2^{141}$	$1.61 \cdot 2^{143}$	2 465	$1.54 \cdot 2^{148}$	$1.94 \cdot 2^{154}$	$\approx 1/e$
this work									
AES-128 (IP MC, $r=1$)	$1.13 \cdot 2^{82}$	$1.32 \cdot 2^{79}$	$1.32 \cdot 2^{77}$	$1.48 \cdot 2^{70}$	$1.08 \cdot 2^{75}$	1665	$1.33 \cdot 2^{82}$	$1.76 \cdot 2^{85}$	$\approx 1/e$
AES-128 (IP MC, $r=2$)	$1.13 \cdot 2^{83}$	$1.32 \cdot 2^{80}$	$1.32 \cdot 2^{78}$	$1.48 \cdot 2^{70}$	$1.08 \cdot 2^{75}$	3329	$1.34 \cdot 2^{83}$	$1.75 \cdot 2^{86}$	≈ 1
AES-192 (IP MC, $r=2$)	$1.27 \cdot 2^{115}$	$1.47 \cdot 2^{112}$	$1.47 \cdot 2^{110}$	$1.47 \cdot 2^{102}$	$1.14 \cdot 2^{107}$	3969	$1.50 \cdot 2^{115}$	$1.11 \cdot 2^{119}$	≈ 1
AES-256 (IP MC, $r=2$)	$1.56 \cdot 2^{147}$	$1.81 \cdot 2^{144}$	$1.81 \cdot 2^{142}$	$1.55 \cdot 2^{134}$	$1.29 \cdot 2^{139}$	4609	$1.84 \cdot 2^{147}$	$1.45 \cdot 2^{151}$	$\approx 1/e$
AES-256 (IP MC, $r=3$)	$1.17 \cdot 2^{148}$	$1.36 \cdot 2^{145}$	$1.36 \cdot 2^{143}$	$1.55 \cdot 2^{134}$	$1.28 \cdot 2^{139}$	6913	$1.38 \cdot 2^{148}$	$1.08 \cdot 2^{152}$	≈ 1

Table: Comparison of cost estimates for Grover's algorithm with $\left\lfloor \frac{\pi}{4} 2^{k/2} \right\rfloor$ AES oracle iterations, minimising *G*-cost (in-place MixColumn beats Maximov's here).

- What can we do about the depth constraint? AES-128 in $\texttt{MAXDEPTH} = 2^{96}$ is the only attack fitting.
- Boyer et al. [BBHT98] propose the following when searching a list of size N:
  1. Run $0.583\sqrt{N}$ Grover iterations.
  2. Measure. If output is wrong, go to 1.
- In theory, the expected number of iterations to win becomes $0.690\sqrt{N} < \frac{\pi}{4}\sqrt{N}$.
- In practice, most often one needs to repeat step 1. at least twice $\implies$ $1.166\sqrt{N} > \frac{\pi}{4}\sqrt{N}$ iterations.

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  - Measure a candidate solution from each machine, and classically check them.
  - Total success probability is $p_S(j) = 1 - (1 - p(j))^S$, where $p(j)$ is the success probability for a single machine.

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  - Measure a candidate solution from each machine, and classically check them.
  - Total success probability is $p_S(j) = 1 - (1 - p(j))^S$, where $p(j)$ is the success probability for a single machine.
  - We want to reduce depth by $\sqrt{S}$. Then, S machines $\implies j = \frac{\pi}{4}\sqrt{\frac{N}{S}}$ iterations. As $S \to \infty$, $p_S(j) \to 0.915$.

- Only chance we have is parallelising. Two strategies, using Kim, Han, and Jeong [KHJ18] nomenclature.
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  - Hence there is no "outer" strategy with $p \approx 1$ that saves $\sqrt{S}$ depth.

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- We want to reduce depth by $\sqrt{S}$. Again, S machines $\implies j = \frac{\pi}{4}\sqrt{\frac{N}{S}}$ iterations. But now, these are the right number of iterations to find the key with $p \approx 1$ in its subset of size N/S!
- In all but one subset we measure a wrong key, in the right subset we measure the correct key. Classically check each, to win with probability $p \approx 1$.

- There's a further advantage of inner parallelisation, when looking for the right key K.
- Take AES-128. We said we need 2 plaintext-ciphertext pairs to uniquely identify $K \in \mathbb{K} = \{0,1\}^{128}$ (i.e. whp no other keys map $m_i \mapsto c_i$ for $i = 1, 2$).

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- Using 1 pair, the probability of only K mapping $m \mapsto c$ exists in $\mathbb{K}$ is $1/e$.
- Now partition $\mathbb{K}$ into S subsets, say $K \in \mathbb{K}_K \subset \mathbb{K}$. The probability that another "spurious" key mapping $m \mapsto c$ exists in $\mathbb{K}_K$ is now smaller than $1 - 1/e$.
- Hence using inner parallelisation increases the success probability of the attack when using 1 plaintext-ciphertext pair. It works similarly for AES-192 and AES-256.

scheme	MD	r	S	$\log_2(\text{SKP})$	D	W	G-cost	DW-cost
AES-128	2^{40}	1	$1.28 \cdot 2^{69}$	-69.36	$1.00 \cdot 2^{40}$	$1.76 \cdot 2^{80}$	$1.07 \cdot 2^{117}$	$1.76 \cdot 2^{120}$
AES-192	2^{40}	1	$1.04 \cdot 2^{133}$	-69.05	$1.00 \cdot 2^{40}$	$1.72 \cdot 2^{144}$	$1.09 \cdot 2^{181}$	$1.72 \cdot 2^{184}$
AES-256	2^{40}	1	$1.12 \cdot 2^{197}$	-69.16	$1.00 \cdot 2^{40}$	$1.08 \cdot 2^{209}$	$1.39 \cdot 2^{245}$	$1.08 \cdot 2^{249}$
AES-128	2^{64}	1	$1.28 \cdot 2^{21}$	-21.36	$1.00 \cdot 2^{64}$	$1.76 \cdot 2^{32}$	$1.07 \cdot 2^{93}$	$1.76 \cdot 2^{96}$
AES-192	2^{64}	1	$1.04 \cdot 2^{85}$	-21.05	$1.00 \cdot 2^{64}$	$1.72 \cdot 2^{96}$	$1.09 \cdot 2^{157}$	$1.72 \cdot 2^{160}$
AES-256	2^{64}	1	$1.12 \cdot 2^{149}$	-21.16	$1.00 \cdot 2^{64}$	$1.08 \cdot 2^{161}$	$1.39 \cdot 2^{221}$	$1.08 \cdot 2^{225}$
AES-128*	2^{96}	2	$1.00 \cdot 2^{0}$	$-\infty$	$1.08 \cdot 2^{75}$	$1.63 \cdot 2^{11}$	$1.34 \cdot 2^{83}$	$1.75 \cdot 2^{86}$
AES-192	2^{96}	2	$1.05 \cdot 2^{21}$	$-\infty$	$1.00 \cdot 2^{96}$	$1.74 \cdot 2^{33}$	$1.09 \cdot 2^{126}$	$1.74 \cdot 2^{129}$
AES-256	2^{96}	2	$1.12 \cdot 2^{85}$	-85.16	$1.00 \cdot 2^{96}$	$1.09 \cdot 2^{98}$	$1.39 \cdot 2^{190}$	$1.09 \cdot 2^{194}$

Table: Cost estimates for *inner* parallelization. r is the number of plaintext-ciphertext pairs used. SKP is the probability that spurious keys are present in $\mathbb{K}_K$. All circuits use Maximov's [Max19] MixColumns (shallower designs have a better advantage when parallelising) except for AES-128 at $\texttt{MAXDEPTH} = 2^{96}$.

Some observations:

- Say a candidate scheme for category 5 does a similar analysis, and the best quantum attack with $\texttt{MAXDEPTH} = 2^{40}$ requires $S = 2^{230}$ G-cost.

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  - Does it not meet the criteria? Nobody is going to build 2^{197} quantum computers anyway, so Grover is not really an attack against AES-256 there.
- Logical qubits better be free. Should we introduce `MAXWIDTH`? What would it mean?
  - Maybe that we try to fit Grover within `MAXWIDTH`, compute the success probability for the resulting attack, and then do the same for candidates ("Cat 5, `MD` 2^{40}, `MW` x means no quantum attack with success prob $\geq 2^{-\cdots}$")?

Finally, we can recompute NIST's table, taking into account inner parallelisation advantages.

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NIST Security Strength Category	source	G-cost for MAXDEPTH 2^{40}	2^{64}	2^{96}	approximation
1 AES-128	[Nat16]	2^{130}	2^{106}	2^{74}	$2^{170}/\texttt{MAXDEPTH}$
	this work	$1.07 \cdot 2^{117}$	$1.07 \cdot 2^{93}$	$^{*}1.34 \cdot 2^{83}$	$\approx 2^{157}/\texttt{MAXDEPTH}$
3 AES-192	[Nat16]	2^{193}	2^{169}	2^{137}	$2^{233}/\texttt{MAXDEPTH}$
	this work	$1.09 \cdot 2^{181}$	$1.09 \cdot 2^{157}$	$1.09 \cdot 2^{126}$	$\approx 2^{221}/\texttt{MAXDEPTH}$
5 AES-256	[Nat16]	2^{258}	2^{234}	2^{202}	$2^{298}/\texttt{MAXDEPTH}$
	this work	$1.39 \cdot 2^{245}$	$1.39 \cdot 2^{221}$	$1.39 \cdot 2^{190}$	$\approx 2^{285}/\texttt{MAXDEPTH}$

Table: *approximation* displays the formula used by NIST in [Nat16] for NIST numbers and a rough approximation that would replace the NIST formula based on our results.

LowMC

- LowMC [ARS+15, ARS+16] is a block cipher family designed for FHE and MPC.
- It is used as part of the Picnic [ZCD+17] submission for post-quantum digital signatures.
- We used the same tools used for AES.

scheme	MD	r	S	$\log_2(\text{SKP})$	D	W	G-cost	DW-cost
LowMC L1	2^{40}	1	$1.40 \cdot 2^{80}$	-80.48	$1.00 \cdot 2^{40}$	$1.08 \cdot 2^{91}$	$1.25 \cdot 2^{123}$	$1.08 \cdot 2^{131}$
LowMC L3	2^{40}	1	$1.83 \cdot 2^{147}$	-147.87	$1.00 \cdot 2^{40}$	$1.06 \cdot 2^{159}$	$1.65 \cdot 2^{190}$	$1.06 \cdot 2^{199}$
LowMC L5	2^{40}	1	$1.08 \cdot 2^{214}$	-214.11	$1.00 \cdot 2^{40}$	$1.61 \cdot 2^{225}$	$1.99 \cdot 2^{256}$	$1.61 \cdot 2^{265}$
LowMC L1	2^{64}	1	$1.40 \cdot 2^{32}$	-32.48	$1.00 \cdot 2^{64}$	$1.08 \cdot 2^{43}$	$1.25 \cdot 2^{99}$	$1.08 \cdot 2^{107}$
LowMC L3	2^{64}	1	$1.83 \cdot 2^{99}$	-99.87	$1.00 \cdot 2^{64}$	$1.06 \cdot 2^{111}$	$1.65 \cdot 2^{166}$	$1.06 \cdot 2^{175}$
LowMC L5	2^{64}	1	$1.08 \cdot 2^{166}$	-166.11	$1.00 \cdot 2^{64}$	$1.61 \cdot 2^{177}$	$1.99 \cdot 2^{232}$	$1.61 \cdot 2^{241}$
LowMC L1	2^{96}	2	$1.00 \cdot 2^{0}$	$-\infty$	$1.18 \cdot 2^{80}$	$1.55 \cdot 2^{11}$	$1.06 \cdot 2^{84}$	$1.83 \cdot 2^{91}$
LowMC L3	2^{96}	1	$1.83 \cdot 2^{35}$	-35.87	$1.00 \cdot 2^{96}$	$1.06 \cdot 2^{47}$	$1.65 \cdot 2^{134}$	$1.06 \cdot 2^{143}$
LowMC L5	2^{96}	1	$1.08 \cdot 2^{102}$	-102.11	$1.00 \cdot 2^{96}$	$1.61 \cdot 2^{113}$	$1.99 \cdot 2^{200}$	$1.61 \cdot 2^{209}$

Table: Cost estimates for parallel Grover key search against LowMC under a depth limit MAXDEPTH with *inner* parallelization.

Further research directions:

- Improve the AES oracle with better S-boxes
  - Sacrificing simulatability, it would be possible to use a compiler based on [GKMR14, ZC19] to automatically synthetise smaller circuits.
  - An orthogonal automatic technique could be to use the classical circuit minimizer by [MSR+19, MSC+19] to attempt to further reduce the linear program components.
- Improve the LowMC design by adopting the approach from [DKP+19].
- Redo the analysis in the surface code setting (it would require new implementations probably, maybe a specific surface-code compiler).

- Take some of the quantum algorithms proposed for the candidates (most use Grover), and do a similar analysis of their quantum component. Do they always/never/sometimes hit `MAXDEPTH`?
- Maybe implementing some of these quantum attacks in Q# could give insight.
- What happens if we introduce `MAXWIDTH`? Or some other bound?
- How do the new oracles impact multi-target attacks? E.g. Banegas and Bernstein [BB17].

Thank you.


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