Quantum Lattice Enumeration in Limited Depth

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- So far, all cryptographically relevant solvers are classical routines
- At least two of these, sieving and enumeration, can be “compiled” into quantum algorithms using black-box methods [LMv13, KMPM19, ANS18, BCSS23]
- While the resulting asymptotic quantum speedups are understood, there’s not a lot of work on their concrete cost; only sieving has been explored [AGPS20]

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- Q. Enum algorithms were first demonstrated by Aono *et al.* [ANS18]; asymptotically, they provide a quadratic speedup
- Our work looks at the "max-depth" setting, where quantum computation is noisy, and long serial computation causes memory to "decohere" [Nat16, Pre18]
- Our results suggest that, as is the case for Grover search against block ciphers [JNRV20], quantum speedups in this setting **may** not apply

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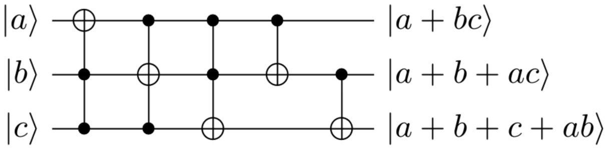

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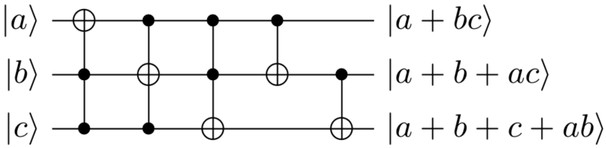

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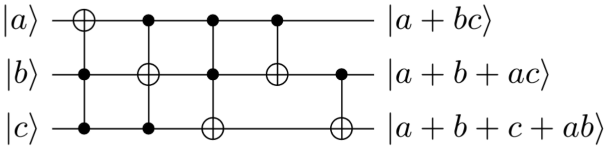

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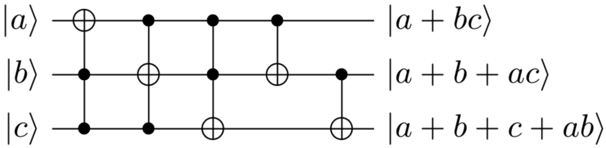

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- The cost of a circuit can be expressed in terms of different metrics, e.g. by counting wires, components, depth, area. . .

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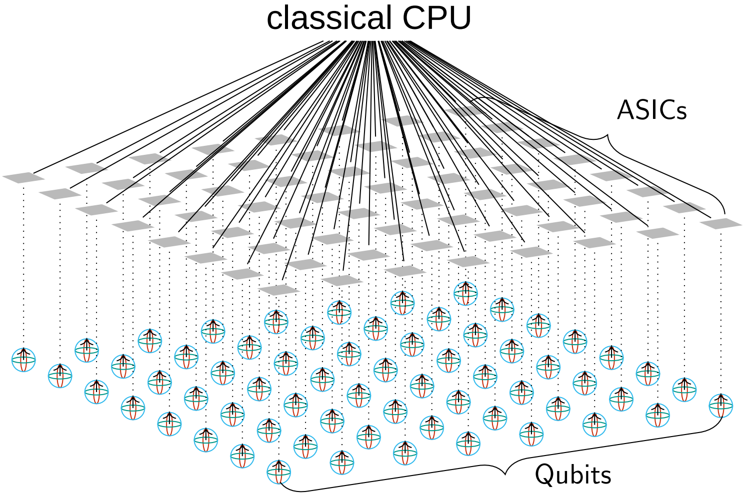

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- $MD = 2^{96} \approx$ "gates that atomic scale qubits with speed of light propagation times could perform in a millennium"

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- Grover search parallelises badly [Zal99], causing the concrete quantum advantage to strongly reduce [JNRV20].

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- As used in lattice reduction, in dimension n, this requires poly(n) memory, and $\mathbb{E}[\# T] = 2^{\frac{1}{8} n \log n + o(n)}$ time on average [ABF+20]

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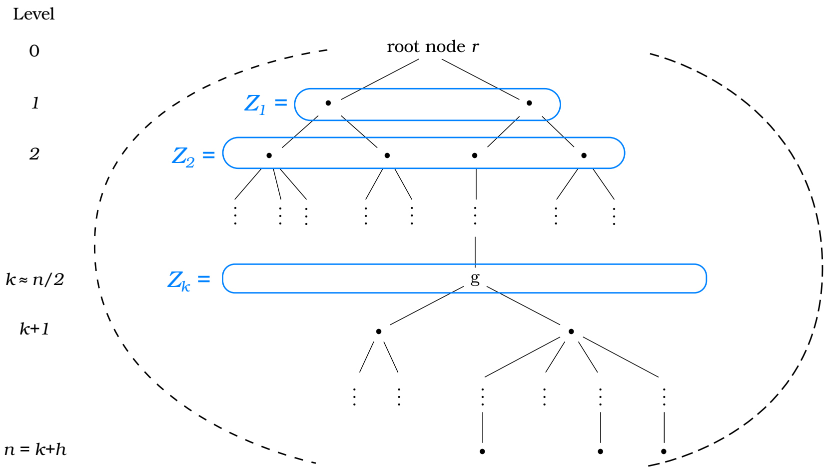
We can see this as searching for a "marked leaf" in a tree, where a leaf is marked if its norm is $\leq R$.

A look at the enumeration tree T



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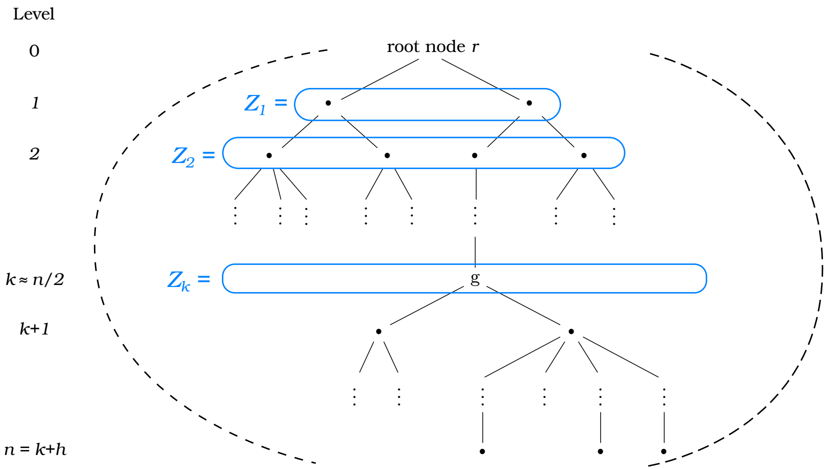

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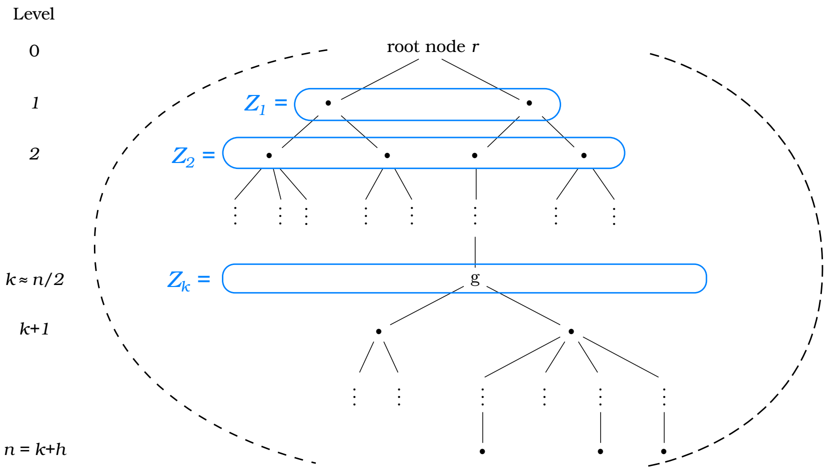

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- The tree size can be somewhat reduced by "pruning" nodes that are unlikely to yield a marked leaf

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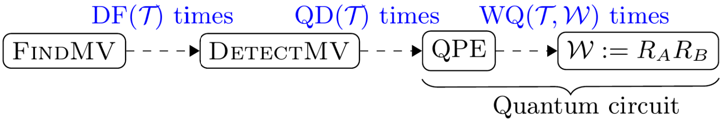
- For trees with one (randomly distributed) marked leaf and $\#T \approx \mathcal{T}$:

  Classical average-case runtime $O(\#T) \mapsto$ quantum average case $\tilde{O}(\sqrt{\#T \cdot n})$

Montanaro's tree search



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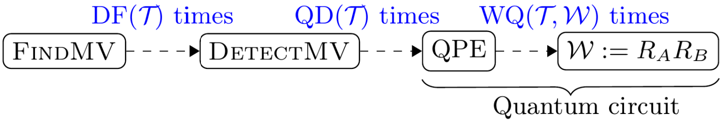

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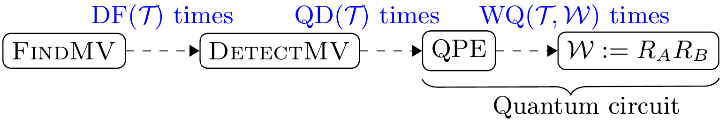

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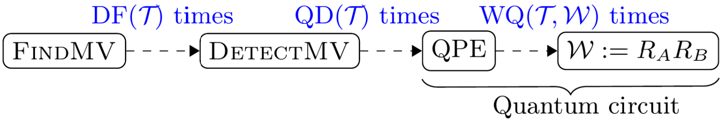

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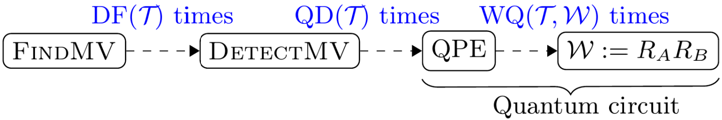

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- Our objective is to lower-bound the gate-cost of FindMV(T), while keeping the serial quantum depth within max-depht MD

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- Finally, we check if the resulting circuit depth of QPE is $\leq MD$

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- I do know Jensen's inequality!
  $$\mathbb{E}[\sqrt{\#T}] \leq \sqrt{\mathbb{E}[\#T]}$$
- Just wait a handful of slides

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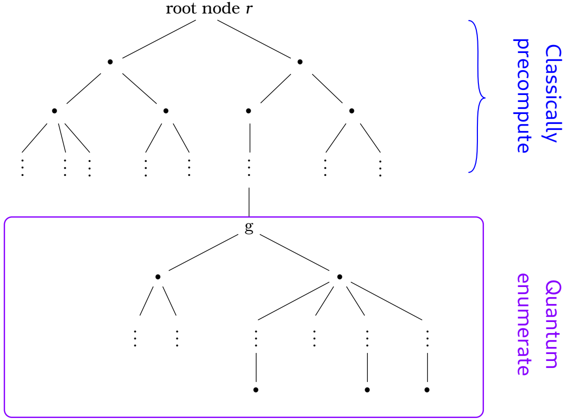

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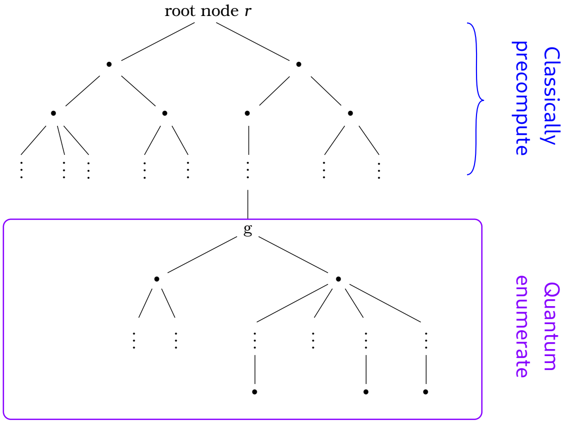
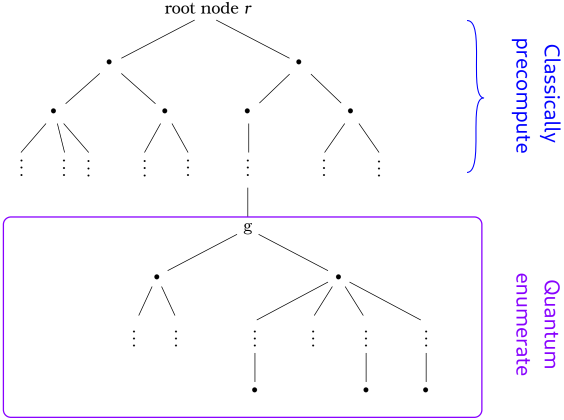
- Precompute nodes up to level $k > 1$, run FindMV on the subtrees.



- We plausibly don't fit within 2^{96} depth
- We need smaller trees to enumerate

Classic trick from parallel enumeration

- Precompute nodes up to level $k > 1$, run FindMV on the subtrees.
- We can estimate the size of subtrees with similar techniques as for the full tree.



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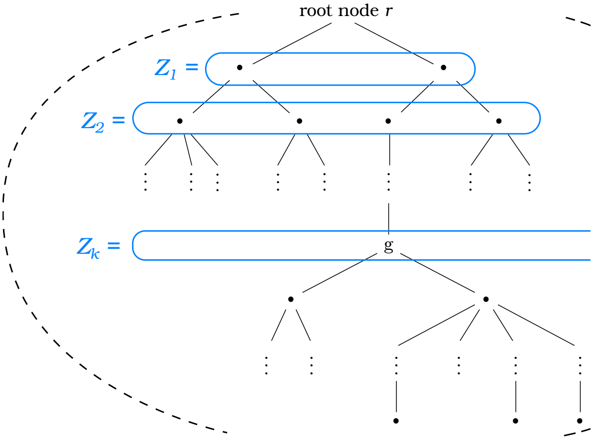

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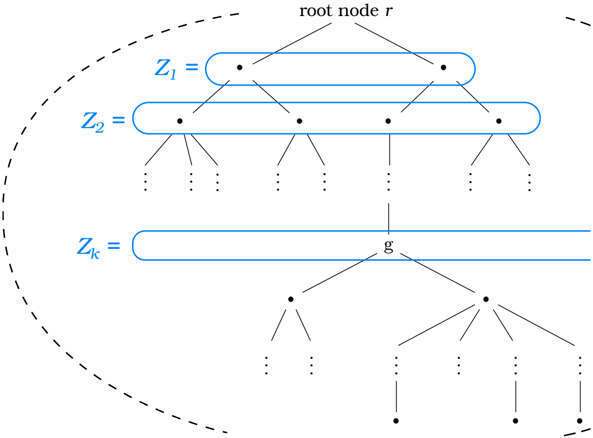

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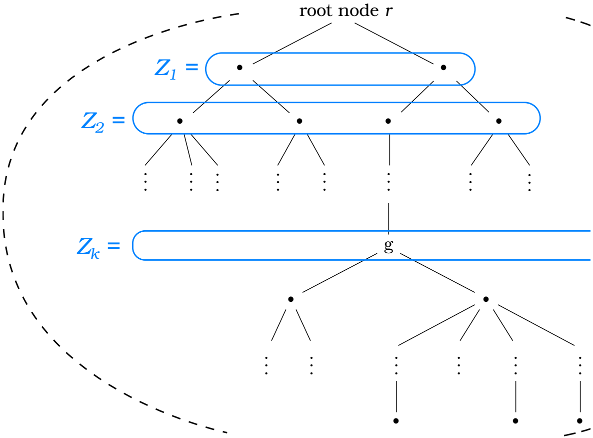

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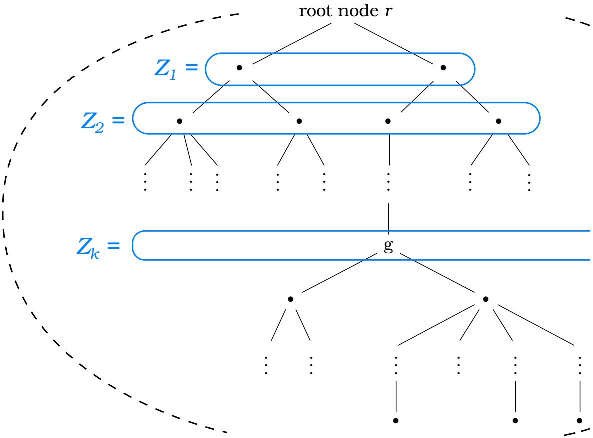

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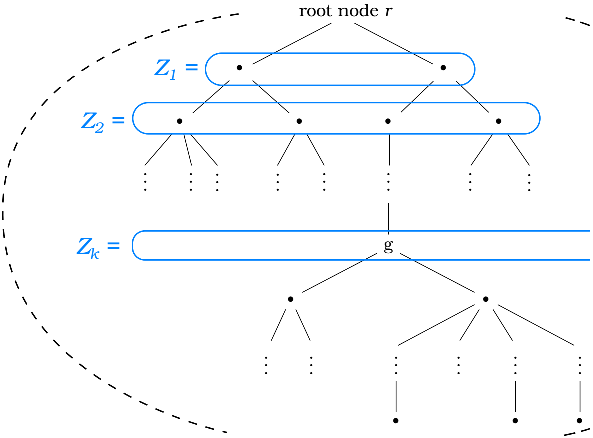

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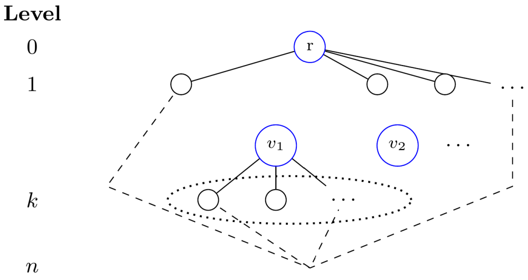

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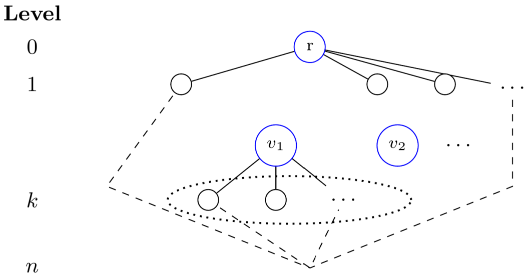

Disclaimer

qRAM (a.k.a. QRACM) may be quite costly to access [JR23]. Yet, many quantum-classical speedups assume it.

One last step: expected square roots

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- Without such bounds, we can run attack cost estimates as a function of z, and see at what point the hypothetical attack becomes viable

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Quantum gate-cost

$$\begin{aligned}\mathop{\mathbb{E}}_{\substack{\text{random}\\ \text{tree } T}}\left[\text{Quantum Gates}\right] &\approx \frac{H_k}{2^y}\cdot \mathbb{E}\left[\mathrm{Gates}(\mathrm{FindMV}(T(g)))\right]\\ &\geq \frac{H_k}{2^y}\cdot \mathbb{E}\left[\sqrt{\# T(v)\cdot(n-k+1)}\right]\cdot \mathrm{Gates}(W)\\ &= \frac{H_k}{2^y}\cdot\frac{1}{2^z}\sqrt{\mathbb{E}\left[\# T(v)\cdot(n-k+1)\right]}\cdot \mathrm{Gates}(W)\end{aligned}$$

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- We report z, k minimising *classical* $+$ *quantum gate-cost*

more likely to be feasible — less likely to be feasible

		$\log \mathbb{E}[\text{GCost}]$ (with $\mathcal{W}$ as in § 4.1) below...			$\log \mathbb{E}[\text{GCost}]$ (with $\mathcal{W}$ as in § 4.2) below...		
MD	Kyber	Target security	Grover on $\text{AES}_{\{128,192,256\}}$	Quasi-Sqrt $^1/_b\sqrt{\#\mathcal{T} \cdot h}$	Target security	Grover on $\text{AES}_{\{128,192,256\}}$	Quasi-Sqrt $^1/_b\sqrt{\#\mathcal{T} \cdot h}$
2^{40}	-512						
	-768						
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2^{64}	-512						
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	-768						
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∞	-512	$z \geq 0,\ k = 0$	$z \geq 9,\ k = 0$	$z \geq 1,\ k = 0$	$z \geq 0,\ k = 0$	$z \geq 33,\ k = 0$	$z \geq 26,\ k = 0$
	-768	$z \geq 0,\ k = 0$	$z \geq 52,\ k = 0$	$z \geq 1,\ k = 0$	$z \geq 1,\ k = 0$	$z > 64$	$z \geq 27,\ k = 0$
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2^{64}	-512						
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2^{96}	-512	$z \geq 0,\ k \leq 58$	$z \geq 8,\ k \leq 53$	$z \geq 1,\ k \leq 58$	$z \geq 0,\ k \leq 63$	$z \geq 33,\ k \leq 54$	$z \geq 25,\ k \leq 58$
	-768	$z \geq 23,\ k \leq 106$	$z \geq 56,\ k \leq 62$	$z \geq 36,\ k \leq 77$	$z \geq 40,\ k \leq 77$	$z > 64$	$z \geq 52,\ k \leq 77$
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	-768	$z \geq 39,\ k \leq 114$	$z \geq 57,\ k \leq 77$	$z \geq 52,\ k \leq 77$	$z \geq 55,\ k \leq 111$	$z > 64$	$z > 64$
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Can we say anything about it?

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- Experimental evidence up to $\beta = 70$ says $z \approx 1$
- We can prove lower bounds on $\mathbb{E}[\sqrt{\#T}]$ based on the additive and multiplicative Jensen's gaps, implying:

$$\mathbb{E}[\sqrt{\#T}] \geq \max\left\{\sqrt{\mathbb{E}[\#T]} - \sqrt[4]{\mathbb{V}[\#T]}, \quad 2^{-\frac{1}{2\ln 2}\sqrt[4]{\mathbb{V}[\#T]}} \cdot \sqrt{\mathbb{E}[\#T]}\right\}.$$

But both depend on $\mathbb{V}[\#T]$.

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- Currently searching for attack costs is an optimisation problem. Can we find a closed formula? This would allow running it as part of "estimator" scripts.
- There quite a few places where our analysis may not be tight, meaning actual costs are likely higher.

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Thank you

Slides @ https://fundamental.domains

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